Creating Voronoi Diagrams Using Delaunay Tetrahedralisations

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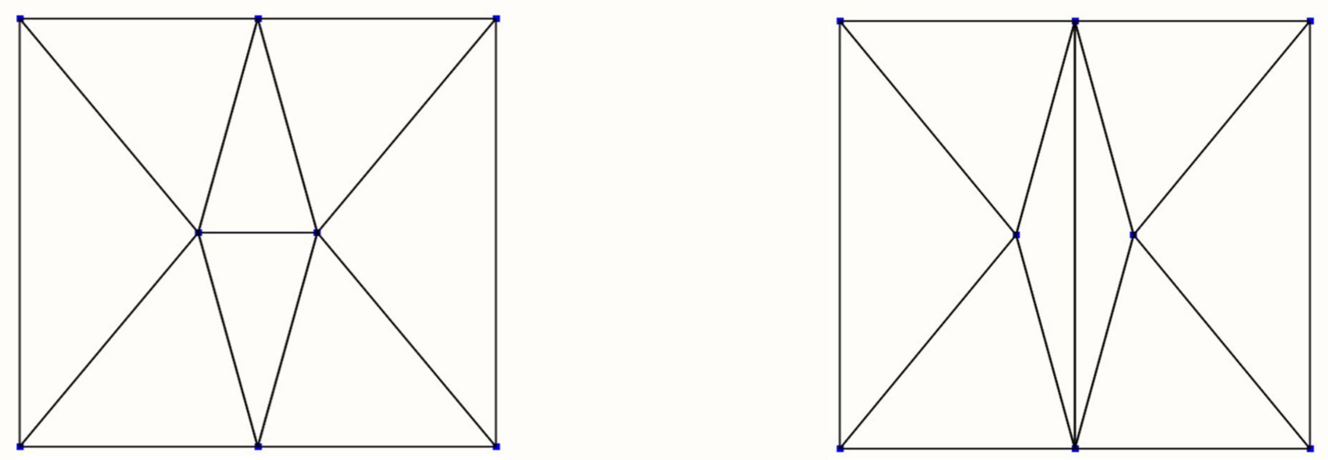
Introduction

The origin of the Voronoi diagram dates back to the 17th century with illustrations in René Descartes' *Principia philosophiae*. Descartes showed a decomposition of space into convex regions, each region consisting of matter revolving around one fixed star. By connecting any two points by an edge if their associated convex regions have boundary in common, the dual structure to the Voronoi diagram is obtained, called the Delaunay triangulation. The basis of this project is to compute a Delaunay triangulation using the Bowyer Watson algorithm and to extract the Voronoi Diagram from this.

Background Definitions

Definition

A *triangulation* of a point set $A \subset \mathbb{R}^d$ is a simplicial d-complex K with vertices A such that the union of all d-simplices in K is the convex hull of A. A *Delaunay triangulation* $DT(A)$ of $A \subset \mathbb{R}^d$ is a triangulation of A such that no point of A lies in an open d-disc whose boundary circumscribes a d-simplex in $DT(A)$. See Figure 1.



(a) Delaunay triangulation of 8 points (b) Non Delaunay triangulation of 8 points

Figure 1: Triangulations of 8 points in the plane.

Definition

Consider a point set $A \subset \mathbb{R}^d$. The *Voronoi region* R_x associated with the point x in A is a possibly unbounded convex d-polytope which consists of those points in $\mathbb{R}^d$ whose distance to x is not greater than their distance to any other point of A.

Definition

The *Voronoi diagram* $V(A)$ induced by A is a decomposition of $\mathbb{R}^d$ into the Voronoi regions associated with the points of A. $V(A)$ will often be referred to as the Voronoi diagram of A.

Remark

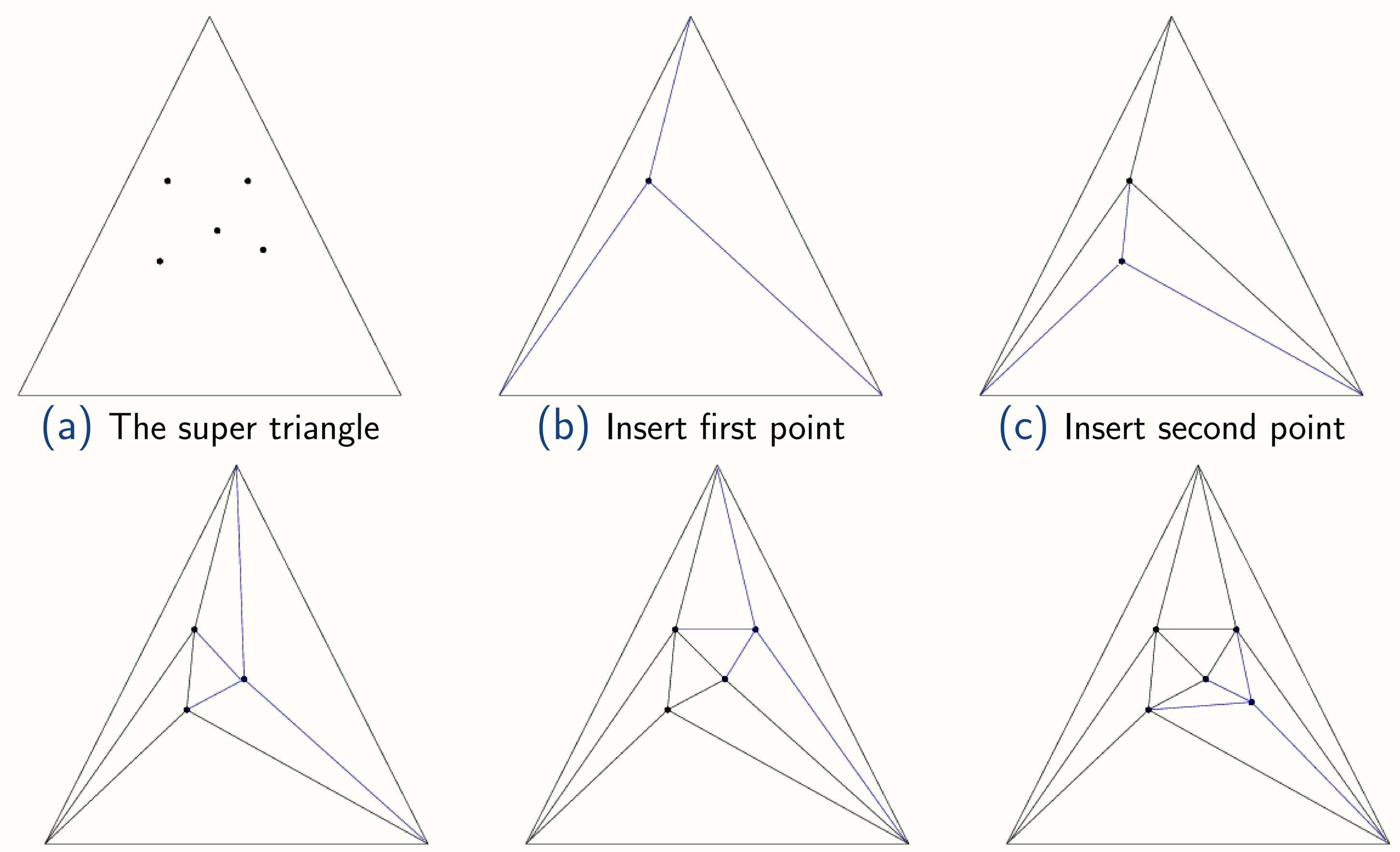
Notice that a Delaunay triangulation of a point set A is not always unique, while the Voronoi diagram of a point set A is unique.

Bowyer Watson Algorithm

The Bowyer Watson algorithm gives an incremental method of producing a Delaunay triangulation of a point set $A \subset \mathbb{R}^d$. The idea is to add one point from A and update the triangulation to remain a Delaunay triangulation at each step. We present pseudocode for the two dimensional case as it generalises readily to higher dimensions. To generalise the algorithm from two dimensions to n-dimensions, replace all instances of the word triangle with n-simplex, edge with $n-1$ face, the polygonal convex set P with an n-polytope and the circumcircle of a triangle with the $(n-1)$-hypersphere circumscribing an n-simplex.

```
//Data: Input point set A, two empty sets of triangles Del and Bad and a polygon P
//Result: Del will be a Delaunay triangulation of A
Triangle_set bowyer_watson(Point_set A) {
  Polygon P; Triangle_set Bad; //Bad holds triangles with non-empty circle
  create a super triangle which contains all points of A and add it to Del;
  for (each point x in A) {
    empty Bad; //Clear the set of bad triangles
    for (each triangle T in Del) { //Find the new bad triangles
      if (x lies inside the circumcircle of T) {
          add T to Bad and remove T from Del;
      }
    }
    for (each triangle T in Bad) { //Find the boundary of the bad triangles
      if (an edge of T is not shared by another triangle in Bad) {
        add that edge to P;
       }
    }
    for (each edge e of P) { //Retriangulate inside P
      form a new triangle by joining e to x and add this triangle to Del;
    }
  }
  Remove all triangles which share a vertex with the super triangle from Del;
  return Del;
}
```

In Figure 2 we present an example of the Bowyer Watson algorithm simulated on five points in $\mathbb{R}^2$. At each step after (a), the new Delaunay edges are shown in blue. Notice that at each step, we add at most two triangles to the triangulation, which is a general result in two dimensions. In Figure 2f, removing all triangles which share a vertex with the super triangle leaves the Delaunay triangulation of the five points.



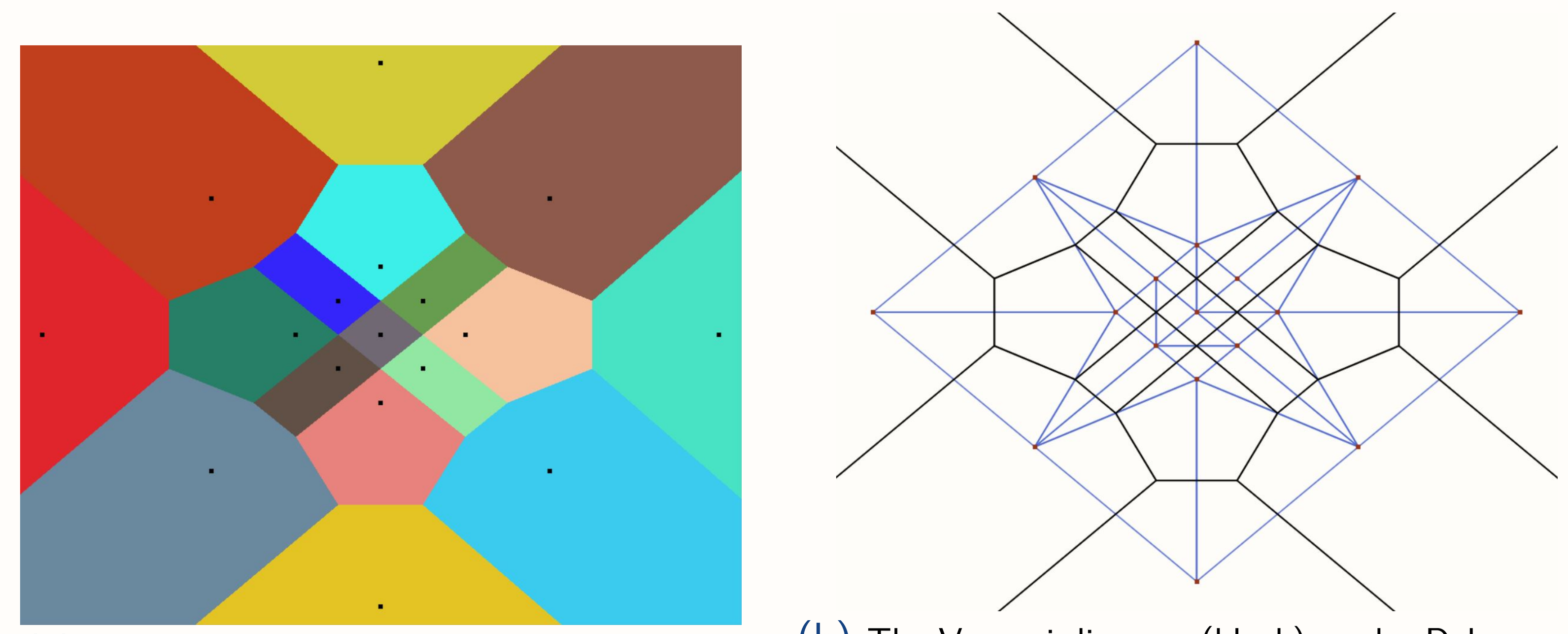
(a) The super triangle (b) Insert first point (c) Insert second point
(d) Insert third point (e) Insert fourth point (f) Insert fifth point

Figure 2: Every step of the Boywer Watson algorithm in our example, bar the last.

Voronoi Diagram Extraction

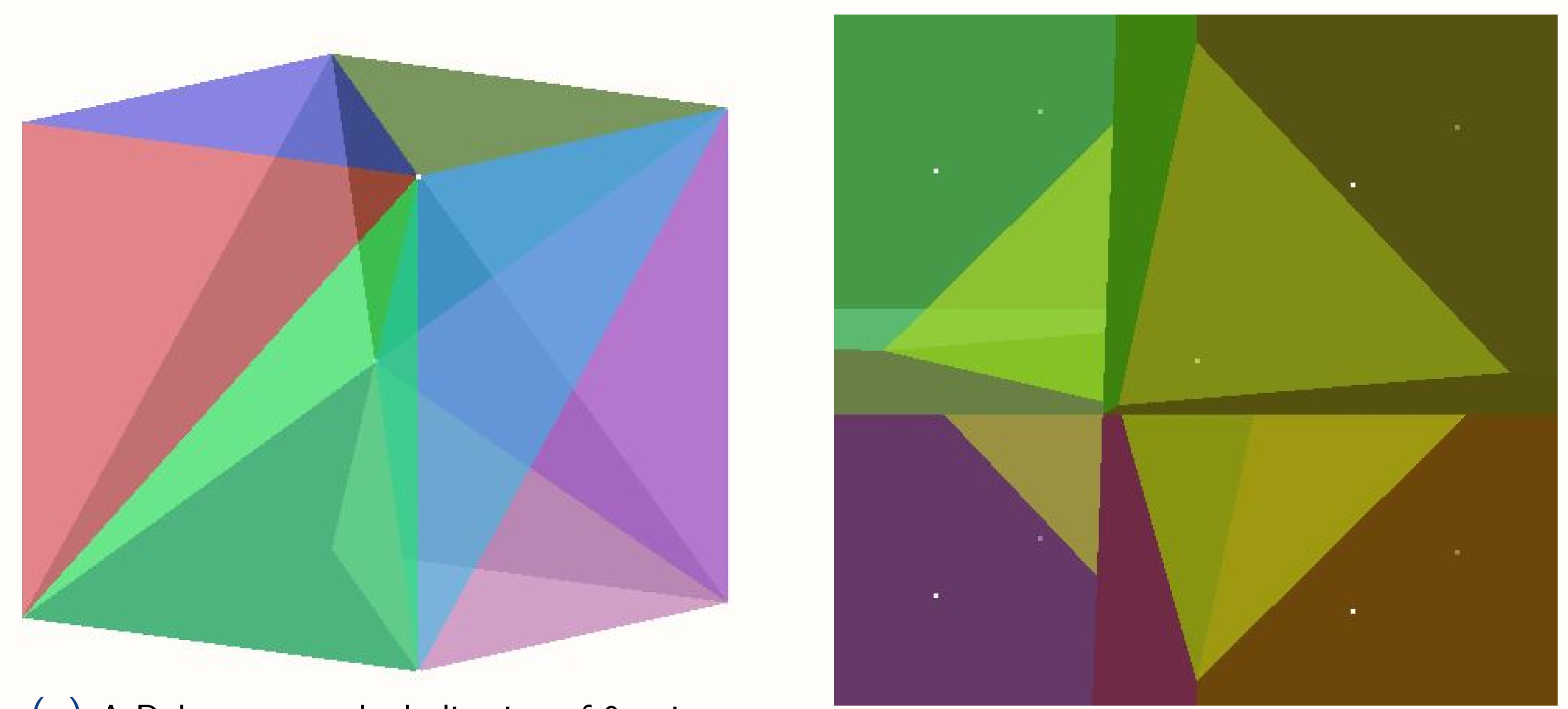
A vertex in the Voronoi diagram or Delaunay triangulation will be called a Voronoi vertex or Delaunay vertex respectively, and similarly for other geometrical structures. Given a point set $A \subset \mathbb{R}^d$ the duality between $V(A)$ and $DT(A)$ is the following. Each n-face of a Voronoi d-polytope (Voronoi region) corresponds to one and only one $d-n$ face of a Delaunay d-simplex (Delaunay triangle). In two dimensions, each Voronoi vertex corresponds to a Delaunay triangle as the circumcenter of the triangle. Each Voronoi edge corresponds to a Delaunay edge as a line segment or a ray of the perpendicular bisector of the edge. Finally, each Voronoi polygon corresponds to a Delaunay vertex v as the polygon formed by all Voronoi edges corresponding to Delaunay edges containing v.

Example Results



(a) The Voronoi diagram induced by 17 points (b) The Voronoi diagram (black), and a Delaunay triangulation (blue) of 17 points

Figure 3: Voronoi diagram and Delaunay triangulation of 17 points in the plane.



(a) A Delaunay tetrahedralisation of 9 points (b) The Voronoi Diagram induced by 9 points

Figure 4: Voronoi diagram and Delaunay tetrahedralisation of 9 points consisting of the vertices of the unit cube and the center of the cube.

Applications

1. Texture generation: The Voronoi Diagram induced by point sets can be used to produce natural textures in computer graphics such as lava like textures, or cobblestone flooring.
2. Geostatistics: Given measurements of the amount of gold at exploratory drill points in a region of mountains, the Voronoi diagram induced by the drill points would give a method of estimating the area in these mountains with the highest concentration of gold deposits.
3. Terrain modelling: Given a set of sample points on the surface of a shape, the Delaunay triangulation could be used to model the full terrain of the shape's surface.

Please feel free to discuss this project with me. In particular improving efficiency of the Bowyer Watson algorithm, programming geometric calculations and further discussion of the three dimensional case are interesting but are not included here due to lack of space.